lec 22

General functions $f = f^+ - f^-$ where $f^\pm$ are positive / negative parts respectively

$$0 \leq f^+ \leq |f| \quad \text{and} \quad 0 \leq f^- \leq |f|, \quad |f| = f^+ + f^-$$

Def: f is integrable if $|f|$ integrable (as a nonneg function)

Prop: $|f|$ integrable $\Leftrightarrow$ f^+ and f^- integrable

$\Rightarrow$ easy. $\Leftarrow$ write $|f| = f^+ + f^-$

Theorem

1) Integral comparison If $|f| \leq g$ and g integrable, then so is f.

$$\left|\int f\right| \leq \int |f| \leq \int g$$

2) If f, g integrable then f, g a.s finite and hence $f+g$ defined a.s.

Therefore $\int \alpha f + \beta g = \alpha \int f + \beta \int g$

where everything is well-defined.

2) Monotonicity $\int f \leq \int g$

3) Additivity over domains

$$\int_{A \cup B} f = \int_A f + \int_B f \quad \text{if } A, B \text{ disjoint}$$

Pf: If $|f| \le g$ Then Take simple $\psi \le |f|$ and we get and $\varphi \le g$. Then monotonicity for simple fns gives

$$\int \psi \le \int \varphi$$ Take sup over φ, ψ to get the result.

$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| \le \left|\int f^+ + \int f^-\right|$$

But we proved linearity of integrals of nonnegative fns using MON.

$$= \left|\int f^+ + f^-\right| = \int |f|$$

2) f, g integrable $\Rightarrow$ $\alpha f + \beta g$ integrable and $\int f+g = \int f + \int g$
(Set $\alpha = \beta = 1$)

$\downarrow$

$\Rightarrow \int |f| + |g| < \infty \Rightarrow \int |f+g| < \infty$ (using $|f+g| \le |f| + |g|$

Linearity for nonneg. fns

and $\int |f| < g$ result we proved above

$\Rightarrow f+g$ is integrable by definition. $\Rightarrow (f+g)^+$ and $(f+g)^-$ are integrable.

$$f+g = (f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$$ pointwise on X

Rearrange to get

$$f+g + f^- + g^- = f^+ + g^+$$

Integrate and apply linearity for nonnegative fns:

$$\int f+g + \int f^- + \int g^- = \int f^+ + \int g^+$$

$$\int f+g = \int f^+ - \int f^- + \int g^+ - \int g^- \quad \text{(rearranging numbers)}$$

$$= \int f + \int g \quad \text{(by definition)}$$

3) follows from 2.

Theorem: (Countable additivity) f integrable over $(X, \mathcal{F}, \mu)$

and $X = \bigsqcup_{n=1}^{\infty} X_n$ then

$$\int_X f = \sum_{n=1}^{\infty} \int_{X_n} f$$

Pf: Follows from linearity, $\sum_{n=1}^{k} \int_{X_n} f \longrightarrow \sum_{n=1}^{\infty} \int_{X_n} f$ / by monotone convergence applied individually to positive and negative parts of f.

Take $f \geq 0$

$$\sum_{n=1}^{k} \int_{X_n} f = \int_{X_n} \sum_{n=1}^{k} f \longrightarrow \int \sum_{n=1}^{\infty} f \mathbb{1}_{X_n} = \int f$$

(just a seq. of numbers)

$$\downarrow$$

$$\sum_{n=1}^{\infty} \int_{X_n} f$$

Theorem (continuity) f integrable on $(X, \mathcal{F}, \mu)$

$X_n \uparrow A$ $\quad \int_A f = \lim_{n \to \infty} \int_{X_n} f$

$X_n \downarrow A$ $\quad \int_A f = \lim_{n \to \infty} \int_{X_n} f$

Enough to show it for nonnegative fns by linearity.

$f \geq 0 \quad f \mathbb{1}_{X_n} \uparrow f$ on A. Apply MON

Theorem If f is bounded, vanishes outside a finite measure set, then f is integrable.

Pf: Trivial: Simply do it for $f \geq 0$ and use the definition

Cor: If $\mathcal{T}$ contains a topology and $f: X \to \mathbb{R}$ is continuous and $\mu(X) < \infty$, then f is integrable.
(it just means f is bounded)

Lebesgue DCT

$\{f_n\} \longrightarrow f$, assume f is measurable (or $(X, \mathcal{F}, \mu)$ is complete), $\exists\, g$ st $|f_n| \leq g$ and $\int_X g < \infty$

Then
$$\int_X f_n \longrightarrow \int_X f$$

Pf: By fatou and comparison both $\{f_n\}$ and f are integrable $\int |f| \leq \underline{\lim} \int |f_n| \leq \int g$

Then $g + f_n \geq 0$ and $g - f_n \geq 0$ and by Fatou

$$\int g + f \leq \int g + \underline{\lim} \int f_n$$

$$\int g - f \leq \underline{\lim} \int (g - f_n) \overset{\text{LIN}}{=} \int g - \overline{\lim} \int f_n$$

$\Rightarrow \overline{\lim} \int f_n \leq \int f \leq \underline{\lim} \int f_n$ using linearity.

UI : $\{f_n\}$ is UI if $\forall \epsilon > 0$ $\exists \delta$ st $\mu(A) < \delta$

$\Rightarrow \int_A |f_n| < \epsilon$

Tight : For $\epsilon > 0$

$\exists X_0 \subset X$ st $\int_{X \setminus X_0} |f_n| < \epsilon$ and $\mu(X_0) < \infty$

Prop : If $\{f_n\} = \{f\}$ then $\{f\}$ is UI and tight

Pf : Assume $f \geq 0$ (simply consider $|f|$ instead)

$\exists \psi$ simple st $0 \leq \psi \leq f$ st $\int f \leq \int \psi + \frac{\epsilon}{2}$

$\int_E f = \int_E f - \psi + \int_E \psi$

Since ψ simple it must be bounded. ~~Thus,~~ $\psi \leq M$

So $\int_E f \leq \frac{\epsilon}{2} + M m(E)$ ~~—~~ Choose $\mu(E)$ small enough

For tightness, note that $\mu(\underbrace{x \mid \psi > 0}_{X_0}) < \infty$

So $\int_{X - X_0} f = \int_{X \setminus X_0} f - \psi \leq \frac{\epsilon}{2}$ (since $\psi = 0$ on $X \setminus X_0$)

Vitali convergence : Suppose $\{f_n\}$ is UI and tight

If 1) $f_n \to f$ a.e

2) f is integrable

Then
$$\int_X f_n \longrightarrow \int_X f$$

Cor: Suppose $\{f_n\}$ is UI and tight. Then

$f_n \to f$ in L^1 iff

$f_n \to f$ in measure.

Pf: Note that $|f_n - f| \leq |f_n| + |f|$

We will control $\int |f_n - f|$ (and thus the L^1 norm)

$$\leq \int |f_n| + \int |f|$$

By tightness $\exists$ X_0 st $\int_{X \setminus X_0} |f_n| + |f| < \epsilon$

By UI for any set $\mu(E) < \delta$ $\int |f_n| + |f| < \epsilon$

By Egoroff on X_0 $\exists$ $X_1 \subset X_0$ st $f_n \to f$ uniformly

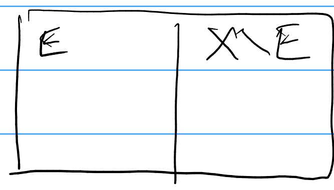
and $\mu(X_0 \setminus X_1) < \delta$

Therefore
$$\int_{X_1} |f_n - f| \leq \mu(X_1) \sup_{x \in X_1} |f_n(x) - f(x)| \to 0$$

So
$$\int_X |f_n - f| = \underbrace{\int_{X_1} |f_n - f|}_{\text{uniform convergence}} + \underbrace{\int_{X_0 \setminus X_1} |f_n - f|}_{\text{UI}} + \underbrace{\int_{X \setminus X_0} |f_n - f|}_{\text{tightness}}$$

Remark: Vitali convergence on general X is different from $\mathbb{R}$ since in general f may not be integrable and has to be assumed.

Ex:



$\mu(E) = \mu(X \setminus E) = \frac{1}{2}$

$\mathcal{F} = \{\emptyset, E, X - E, X\}$

f_n is UI and f_n is tight

f_n is UI since you choose $\delta < \frac{1}{2}$ and there is ONLY one $\emptyset$ so $\mu(\emptyset) < \delta$, so it trivial.

f_n is tight since $\mu(X) = 1$.

$$f_n \longrightarrow f = \begin{cases} -\infty & \text{on } X \setminus E \\ +\infty & \text{on } E \end{cases}$$